Motion Models (cont)

Computing the Density

- to compute

  $\text{prob}(v - \hat{v}, \alpha_1 v^2 + \alpha_2 \omega^2)$,
  $\text{prob}(\omega - \hat{\omega}, \alpha_3 v^2 + \alpha_4 \omega^2)$, and
  $\text{prob}(\hat{\gamma}, \alpha_5 v^2 + \alpha_6 \omega^2)$

  use the appropriate probability density function; i.e., for zero-mean Gaussian noise:

$$\text{prob}(a, b^2) = \frac{1}{\sqrt{2\pi b^2}} e^{-\frac{1 a^2}{2 b^2}}$$

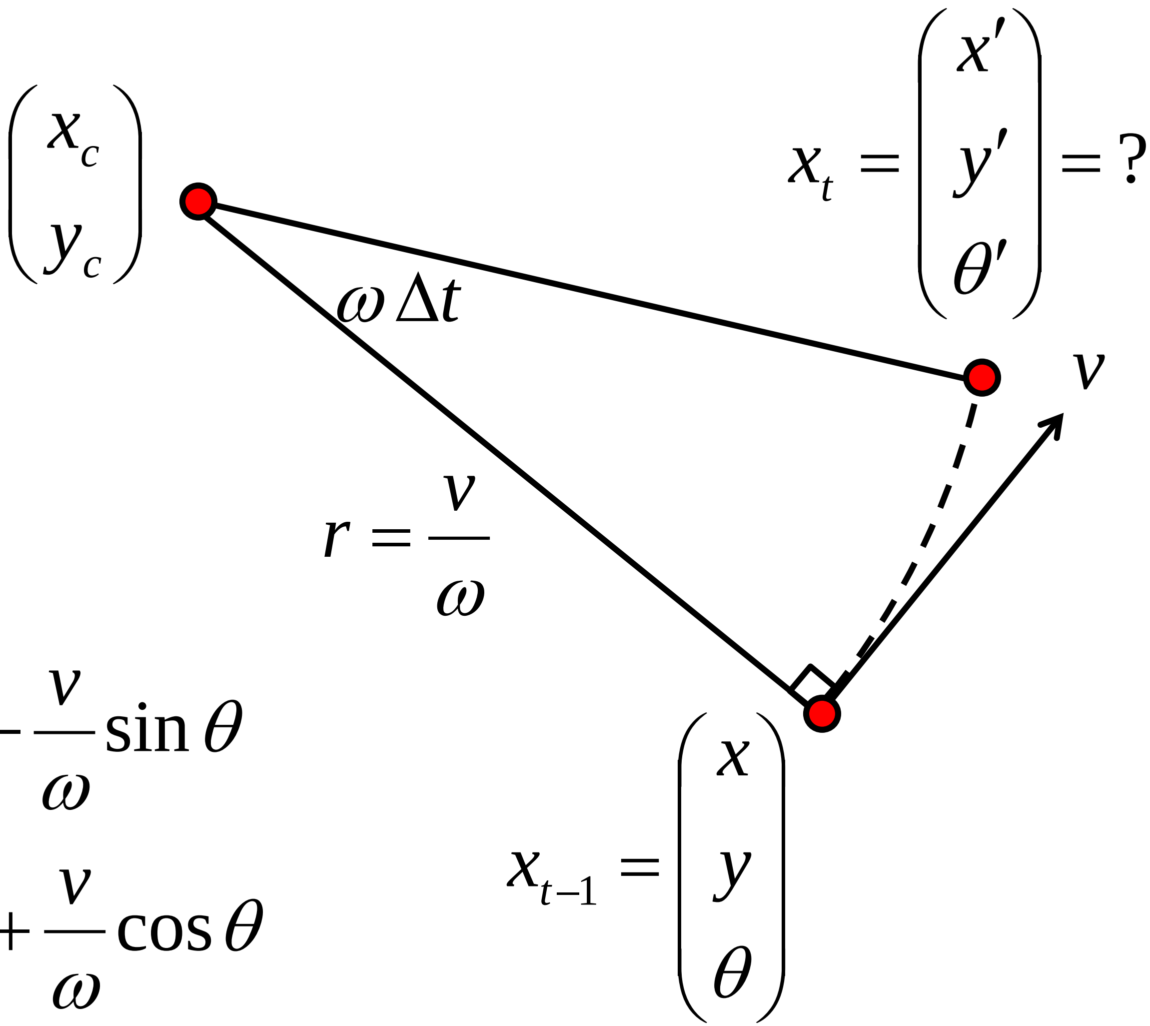
Sampling from the Velocity Motion Model

- suppose that a robot has a map of its environment and it needs to find its pose in the environment
  - this is the robot localization problem
  - several variants of the problem
    - the robot knows where it is initially
    - the robot does not know where it is initially
    - kidnapped robot: at any time, the robot can be teleported to another location in the environment
- a popular solution to the localization problem is the particle filter
  - uses simulation to sample the state density $p(x_t \mid u_t, x_{t-1})$

Sampling from the Velocity Motion Model

- sampling the conditional density is easier than computing the density because we only require the forward kinematics model
  - given the control u_t and the previous pose $x_{t\text{-}1}$ find the new pose x_t

Sampling from the Velocity Motion Model



$$x_c = x - \frac{v}{\omega} \sin\theta$$

$$y_c = y + \frac{v}{\omega} \cos\theta$$

Eqs 5.7, 5.8

Sampling from the Velocity Motion Model

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} x_c + \frac{v}{\omega}\sin(\theta + \omega\Delta t) \\ y_c - \frac{v}{\omega}\cos(\theta + \omega\Delta t) \\ \theta + \omega\Delta t \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v}{\omega}\sin\theta + \frac{v}{\omega}\sin(\theta + \omega\Delta t) \\ \frac{v}{\omega}\cos\theta - \frac{v}{\omega}\cos(\theta + \omega\Delta t) \\ \omega\Delta t \end{pmatrix}$$

Eqs 5.9

*we already derived this for the differential drive!

Sampling from the Velocity Motion Model

- as with the original motion model, we will assume that given noisy velocities the robot can also make a small rotation in place to determine the final orientation of the robot

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{\hat{v}}{\hat{\omega}}\sin\theta + \frac{\hat{v}}{\hat{\omega}}\sin(\theta + \hat{\omega}\Delta t) \\ \frac{\hat{v}}{\hat{\omega}}\cos\theta - \frac{\hat{v}}{\hat{\omega}}\cos(\theta + \hat{\omega}\Delta t) \\ \hat{\omega}\Delta t + \hat{\gamma}\Delta t \end{pmatrix}$$

Sampling from the Velocity Motion Model

Algorithm sample_motion_model_velocity(u_t, x_{t-1})**:**

$\hat{v} = v + \textbf{sample}(\alpha_1\, v^2 + \alpha_2\, \omega^2)$

$\hat{\omega} = \omega + \textbf{sample}(\alpha_3\, v^2 + \alpha_4\, \omega^2)$

$\hat{\gamma} = \textbf{sample}(\alpha_5\, v^2 + \alpha_6\, \omega^2)$

$x' = x - \frac{\hat{v}}{\hat{\omega}} \sin\theta + \frac{\hat{v}}{\hat{\omega}}\, sin(\theta + \hat{\omega}\Delta t)$

$y' = y + \frac{\hat{v}}{\hat{\omega}} \cos\theta - \frac{\hat{v}}{\hat{\omega}}\, cos(\theta + \hat{\omega}\Delta t)$

$\theta' = \theta + \hat{\omega}\Delta t + \hat{\gamma}\Delta t$

return $x_t = (x', y', \theta')^T$

Sampling from the Velocity Motion Model

- the function $\mathrm{sample}(b^2)$ generates a random sample from a zero-mean distribution with variance b^2
- Matlab is able to generate random numbers from many different distributions
  - help randn
  - help stats

How to Sample from Normal or Triangular Distributions?

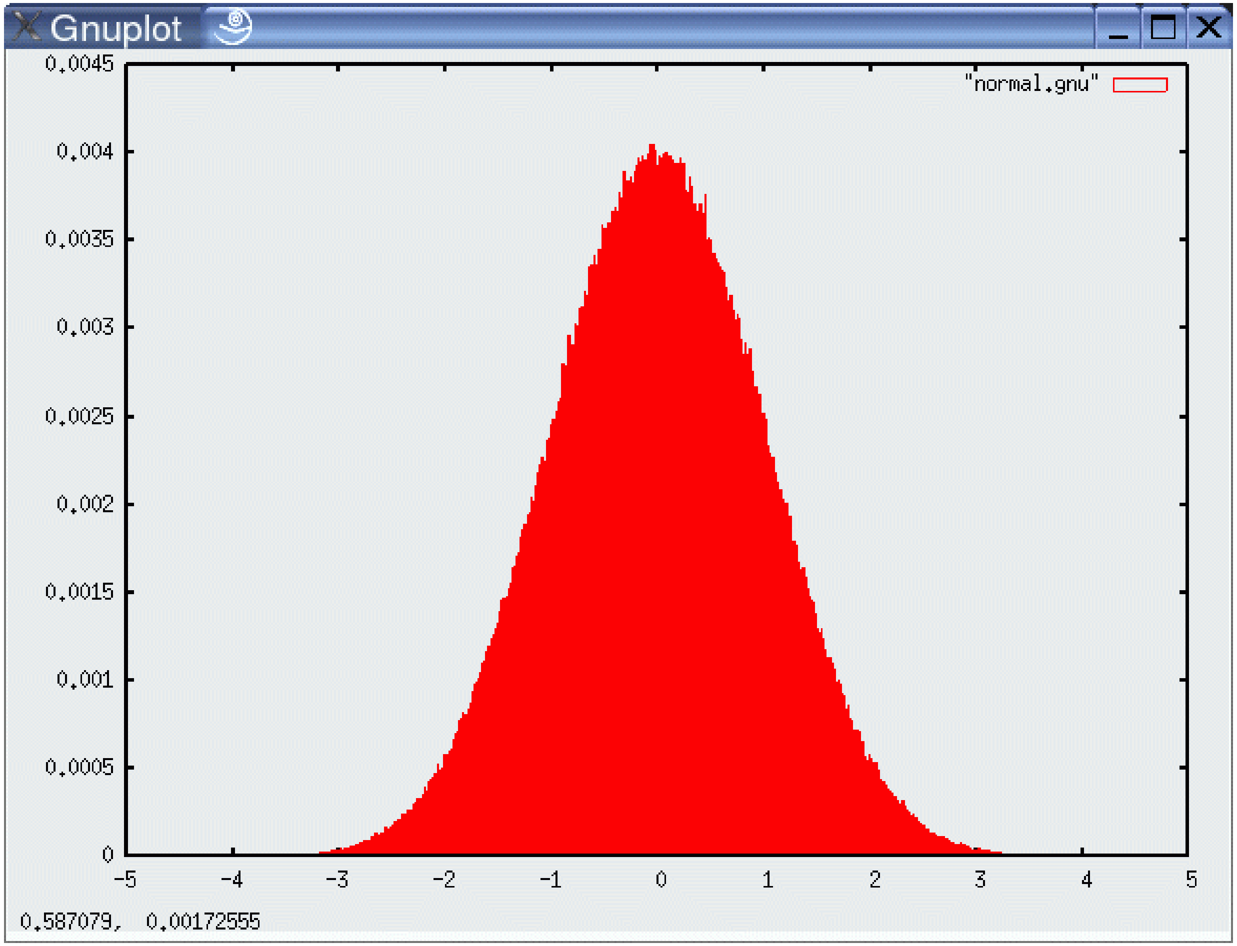
- Sampling from a normal distribution

1. Algorithm **sample_normal_distribution**(b):
2. return $$\frac{1}{2}\sum_{i=1}^{12} rand(-b, b)$$

- Sampling from a triangular distribution

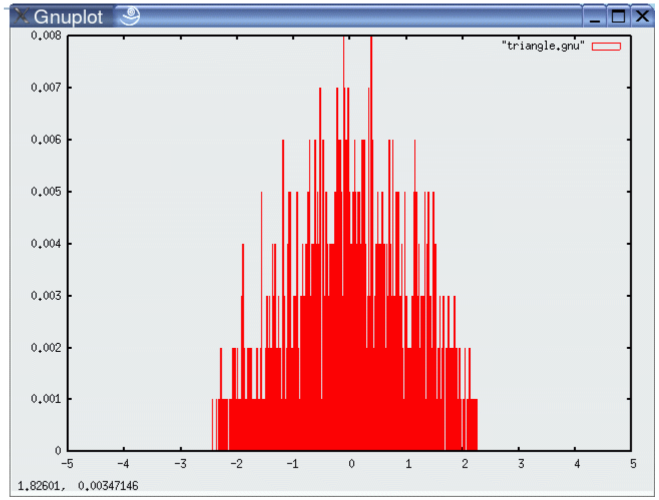
1. Algorithm **sample_triangular_distribution**(b):
2. return $$\frac{\sqrt{6}}{2}\left[\mathbf{rand}(-b, b) + \mathbf{rand}(-b, b)\right]$$

Normally Distributed Samples

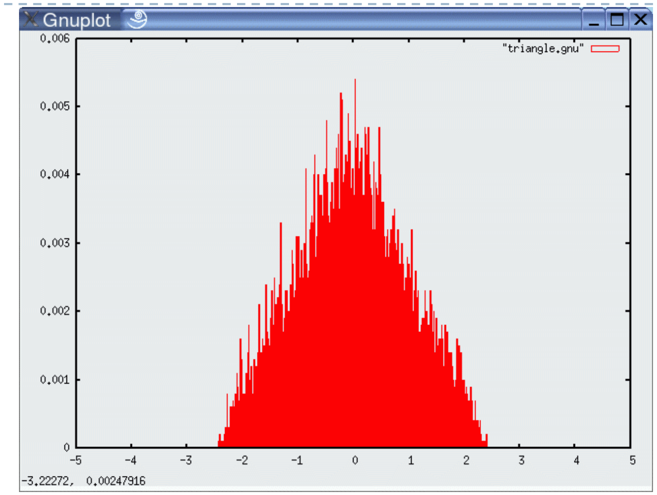


10^6 samples

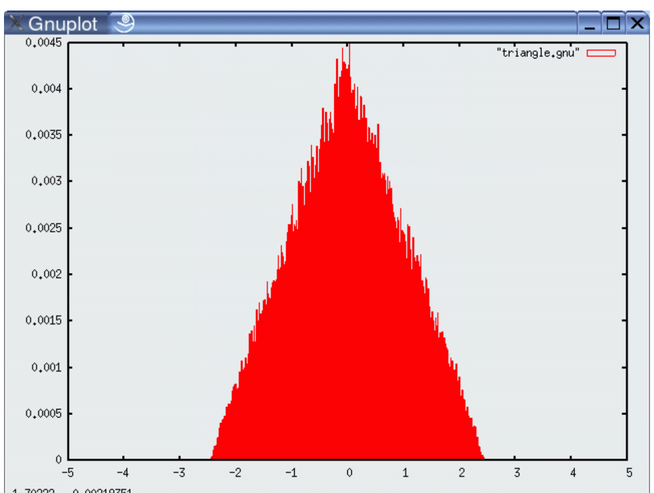
For Triangular Distribution



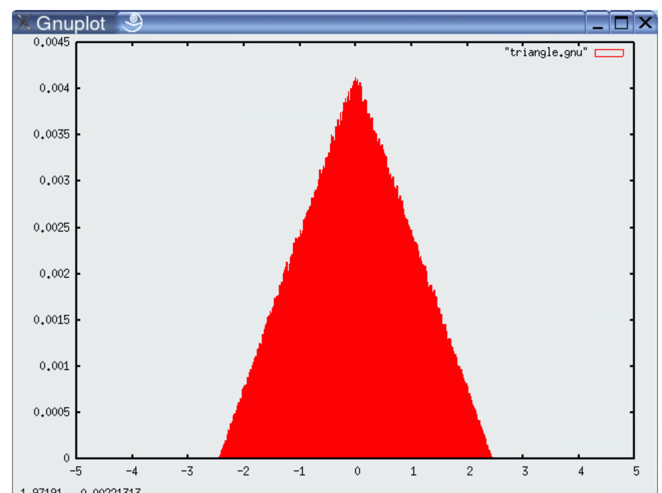
10^3 samples



10^4 samples



10^5 samples



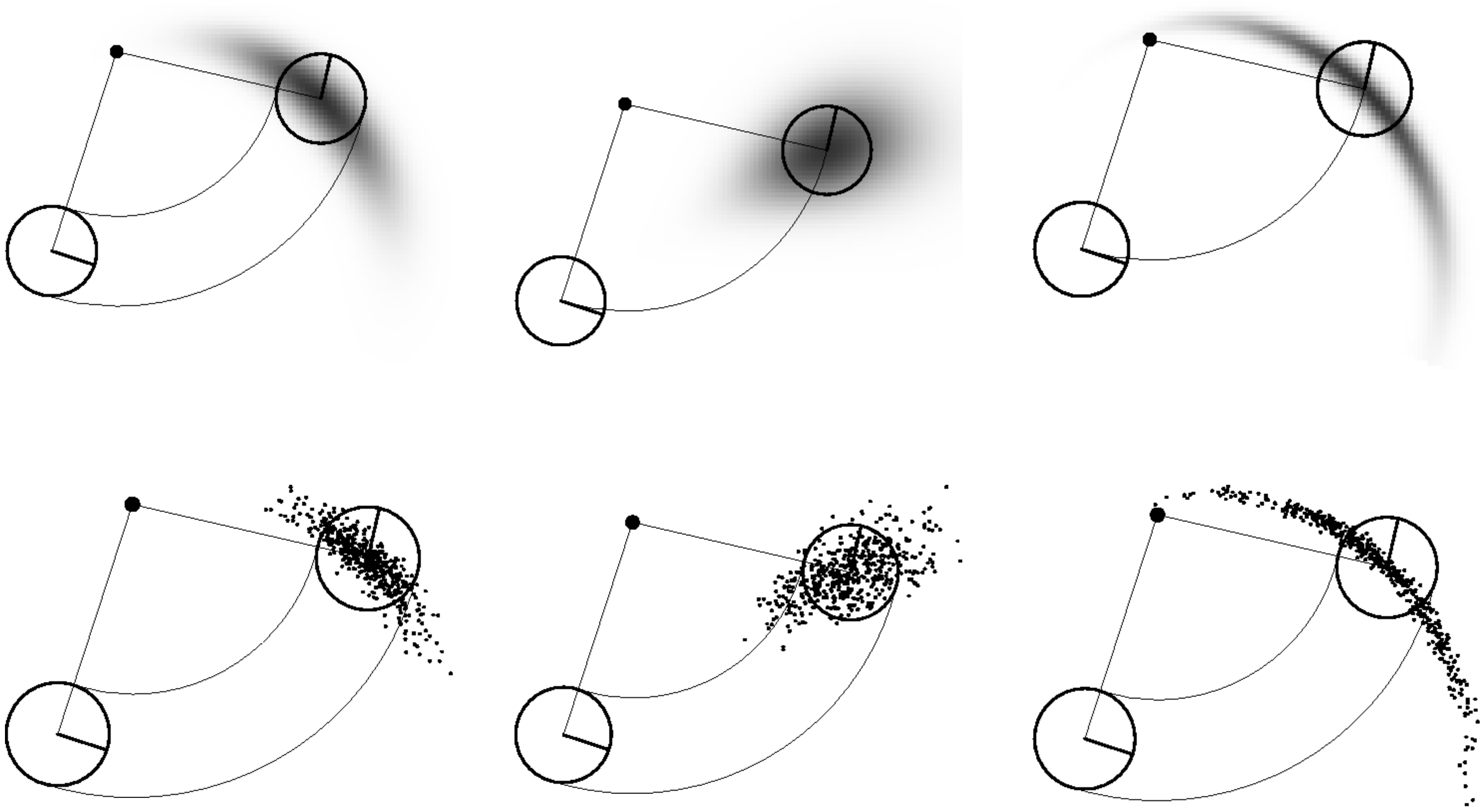
10^6 samples

Rejection Sampling

- Sampling from arbitrary distributions

1. Algorithm **sample_distribution**(*f*,*b*):
2. repeat
3. $x = \mathbf{rand}(-b, b)$
4. $y = \mathbf{rand}(0, \max\{f(x) \mid x \in (-b, b)\})$
5. until ($y \leq f(x)$
6. return x

Examples

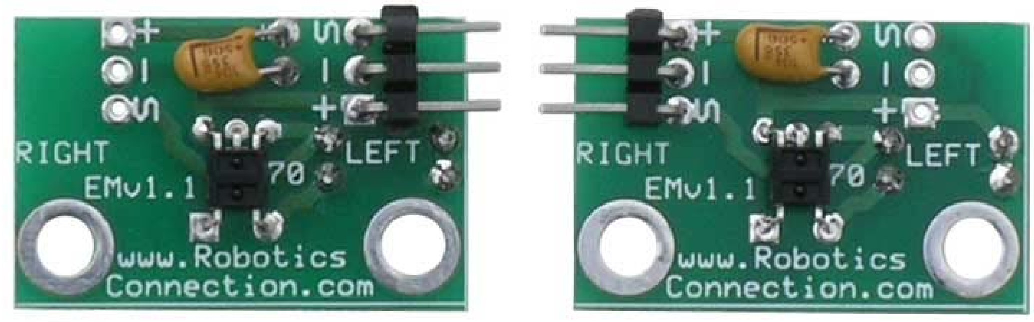


Odometry Motion Model

- many robots make use of odometry rather than velocity
- odometry uses a sensor or sensors to measure motion to estimate changes in position over time
- typically more accurate than velocity motion model, but measurements are available only after the motion has been completed
- technically a measurement rather than a control
  - but usually treated as control to simplify the modeling
- odometry allows a robot to estimate its pose
  - but no fixed mapping from odometer coordinates and world coordinates
- in wheeled robots the sensor is often a rotary encoder

Example Wheel Encoders

These modules require +5V and GND to power them, and provide a 0 to 5V output. They provide +5V output when they "see" white, and a 0V output when they "see" black.

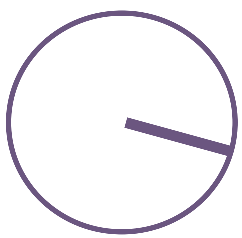


These disks are manufactured out of high quality laminated color plastic to offer a very crisp black to white transition. This enables a wheel encoder sensor to easily see the transitions.

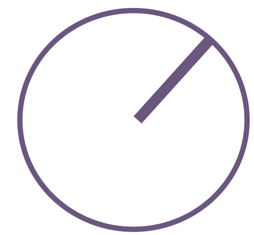
Source: http://www.active-robots.com/

Odometry Model

- when using odometry, the robot keeps an internal estimate of its pose at all time
  - for example, consider a robot moving from pose $\bar{x}_{t-1}$ to $\bar{x}_t$



$$\bar{x}_{t-1} = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{\theta} \end{pmatrix}$$



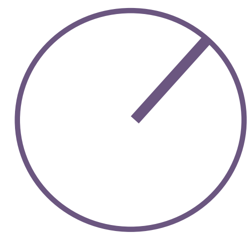
$$\bar{x}_t = \begin{pmatrix} \bar{x}' \\ \bar{y}' \\ \bar{\theta}' \end{pmatrix}$$

Note: bar indicates values in the robot's internal coordinate system

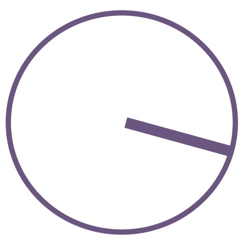
Odometry Model

- the internal pose estimates $\bar{x}_{t-1}$ to $\bar{x}_t$ are treated as the control inputs to the robot:

$$u_t = \begin{pmatrix} \bar{x}_{t-1} \\ \bar{x}_t \end{pmatrix}$$



$$\bar{x}_t = \begin{pmatrix} \bar{x}' \\ \bar{y}' \\ \bar{\theta}' \end{pmatrix}$$

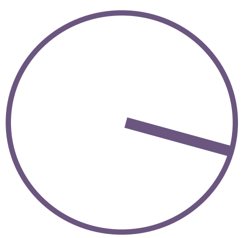


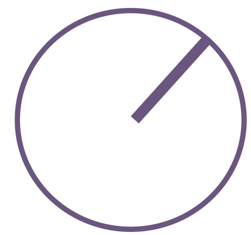
$$\bar{x}_{t-1} = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{\theta} \end{pmatrix}$$

Note: bar indicates values in the robot's internal coordinate system

Odometry Model

- we require a model of how the robot moves from $\bar{x}_{t-1}$ to $\bar{x}_t$
  - there are an infinite number of possible motions between $\bar{x}_{t-1}$ to $\bar{x}_t$





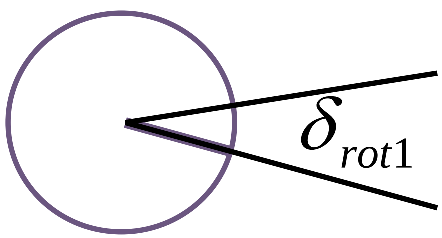
$$\bar{x}_{t-1} = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{\theta} \end{pmatrix}$$

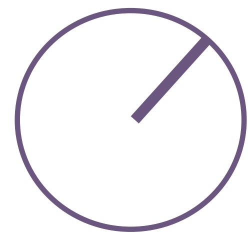
$$\bar{x}_t = \begin{pmatrix} \bar{x}' \\ \bar{y}' \\ \bar{\theta}' \end{pmatrix}$$

Note: bar indicates values in the robot's internal coordinate system

Odometry Model

- assume the motion is accomplished in 3 steps:
  1. rotate in place by δ_{rot1}





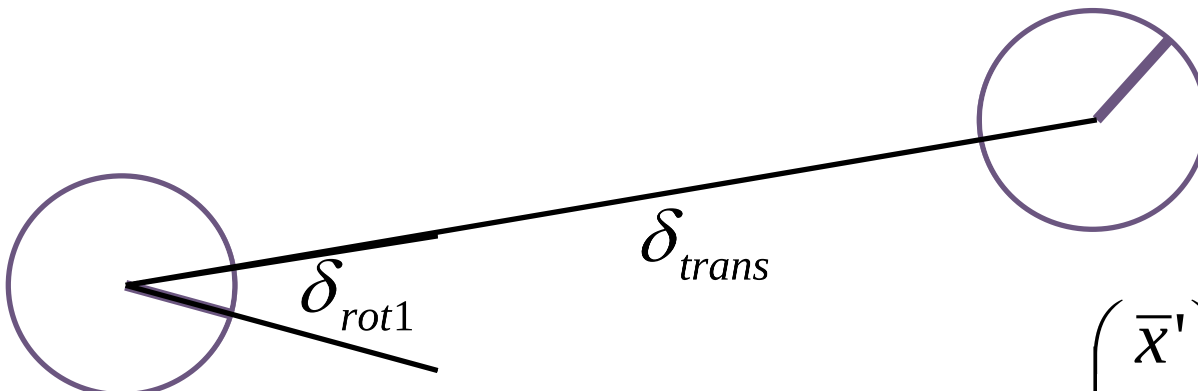
$$\bar{x}_{t-1} = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{\theta} \end{pmatrix}$$

$$\bar{x}_t = \begin{pmatrix} \bar{x}' \\ \bar{y}' \\ \bar{\theta}' \end{pmatrix}$$

Note: bar indicates values in the robot's internal coordinate system

Odometry Model

- assume the motion is accomplished in 3 steps:
  1. rotate in place by δ_{rot1}
  2. move in a straight line by δ_{trans}

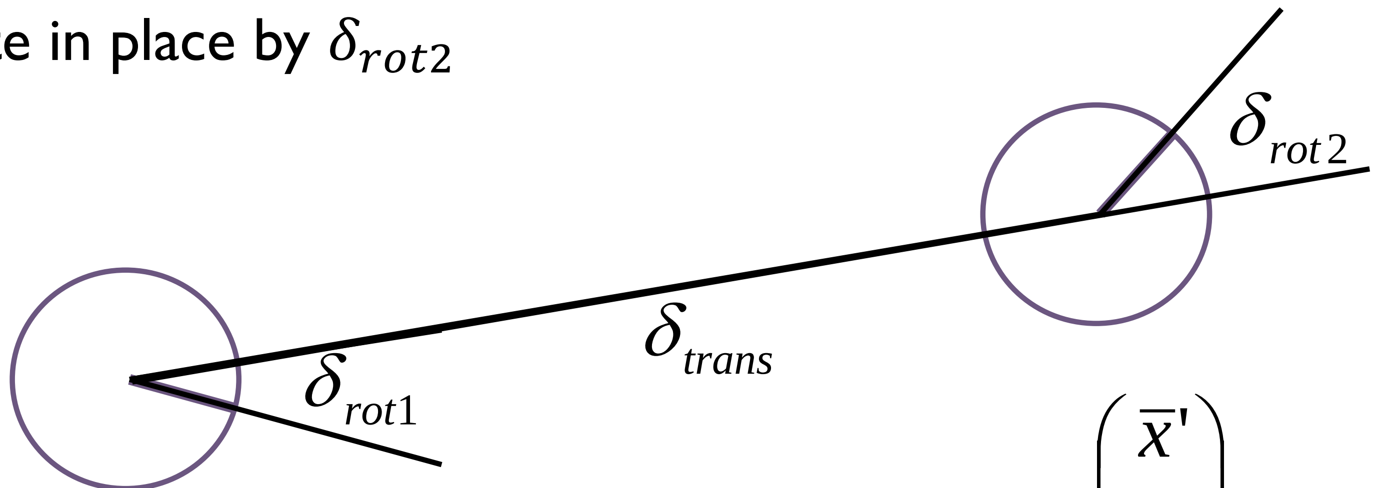


$$\bar{x}_{t-1} = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{\theta} \end{pmatrix} \qquad \bar{x}_t = \begin{pmatrix} \bar{x}' \\ \bar{y}' \\ \bar{\theta}' \end{pmatrix}$$

Note: bar indicates values in the robot's internal coordinate system

Odometry Model

- assume the motion is accomplished in 3 steps:
  1. rotate in place by δ_{rot1}
  2. move in a straight line by δ_{trans}
  3. rotate in place by δ_{rot2}



$$\bar{x}_t = \begin{pmatrix} \bar{x}' \\ \bar{y}' \\ \bar{\theta}' \end{pmatrix}$$

$$\bar{x}_{t-1} = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{\theta} \end{pmatrix}$$

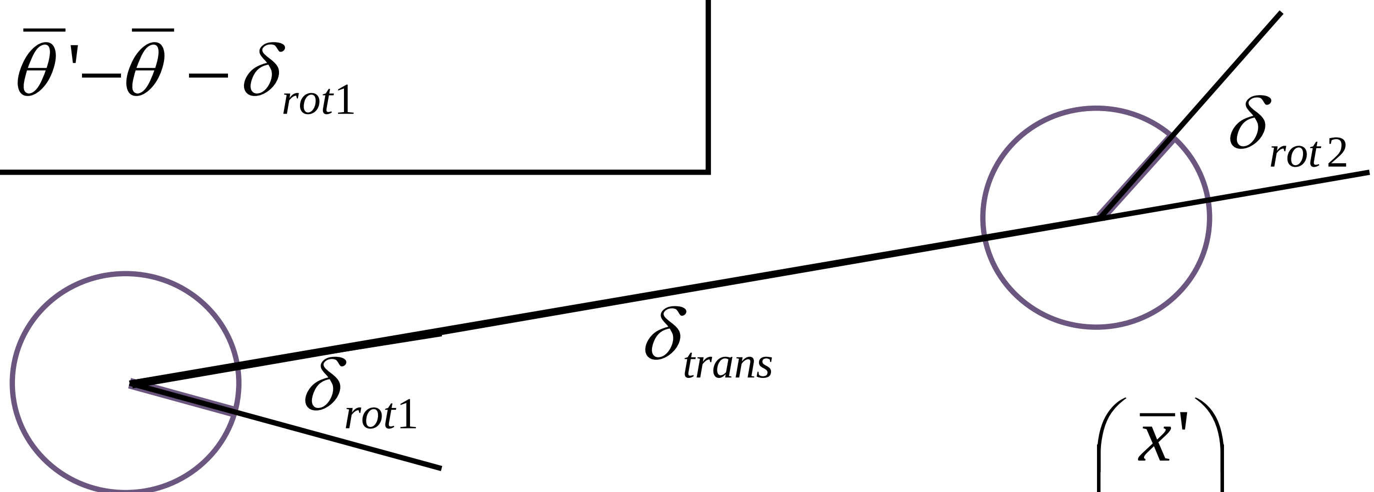
Note: bar indicates values in the robot's internal coordinate system

Odometry Model

$$\delta_{trans} = \sqrt{(\bar{x}' - \bar{x})^2 + (\bar{y}' - \bar{y})^2}$$

$$\delta_{rot1} = \operatorname{atan2}(\bar{y}' - \bar{y}, \bar{x}' - \bar{x}) - \bar{\theta}$$

$$\delta_{rot2} = \bar{\theta}' - \bar{\theta} - \delta_{rot1}$$



$$\bar{x}_t = \begin{pmatrix} \bar{x}' \\ \bar{y}' \\ \bar{\theta}' \end{pmatrix}$$

$$\bar{x}_{t-1} = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{\theta} \end{pmatrix}$$

Note: bar indicates values in the robot's internal coordinate system

Noise Model for Odometry

- the difference between the true motion of the robot and the odometry motion is assumed to be a zero-mean random value

$$\delta_{rot1} - \hat{\delta}_{rot1} = \varepsilon_{\alpha_1 \delta_{rot1}^2 + \alpha_2 \delta_{trans}^2}$$

$$\delta_{trans} - \hat{\delta}_{trans} = \varepsilon_{\alpha_3 \delta_{trans}^2 + \alpha_4 (\delta_{rot1}^2 + \delta_{rot2}^2)}$$

$$\delta_{rot2} - \hat{\delta}_{rot2} = \varepsilon_{\alpha_1 \delta_{rot2}^2 + \alpha_2 \delta_{trans}^2}$$

Sampling from the Odometry Motion Model

- suppose you are given the previous pose of the robot in world coordinates (x_{t-1}) and the most recent odometry from the robot (u_t)
- how do you generate a random sample of the current pose of the robot in world coordinates (x_t)?
  1. use odometry to compute motion parameters $\delta_{rot1}, \delta_{trans}, \delta_{rot2}$
  2. use noise model to generate random true motion parameters $\hat{\delta}_{rot1}, \hat{\delta}_{trans}, \hat{\delta}_{rot2}$
  3. use random true motion parameters to compute a random x_t

Sample Odometry Motion Model

1. Algorithm **sample_motion_model**(u_t, x_{t-1}):
2. $\delta_{rot1} = \text{atan2}(\bar{y}'-\bar{y}, \bar{x}'-\bar{x}) - \bar{\theta}$
3. $\delta_{trans} = \sqrt{(\bar{x}'-\bar{x})^2 + (\bar{y}'-\bar{y})^2}$
4. $\delta_{rot2} = \bar{\theta}' - \bar{\theta} - \delta_{rot1}$
5. $\hat{\delta}_{rot1} = \delta_{rot1} - sample(\alpha_1\ \delta_{rot1}^2 + \alpha_2\ \delta_{trans}^2)$
6. $\hat{\delta}_{trans} = \delta_{trans} - sample(\alpha_3\ \delta_{trans}^2 + \alpha_4\ (\delta_{rot1}^2 + \delta_{rot2}^2))$
7. $\hat{\delta}_{rot2} = \delta_{rot2} - sample(\alpha_1\ \delta_{rot2}^2 + \alpha_2\ \delta_{trans}^2)$
8. $x' = x + \hat{\delta}_{trans} \cos\left(\theta + \hat{\delta}_{rot1}\right)$
9. $y' = y + \hat{\delta}_{trans} \sin\left(\theta + \hat{\delta}_{rot1}\right)$
10. $\theta' = \theta + \hat{\delta}_{rot1} + \hat{\delta}_{rot2}$
11. return $[x' \quad y' \quad \theta']^T$

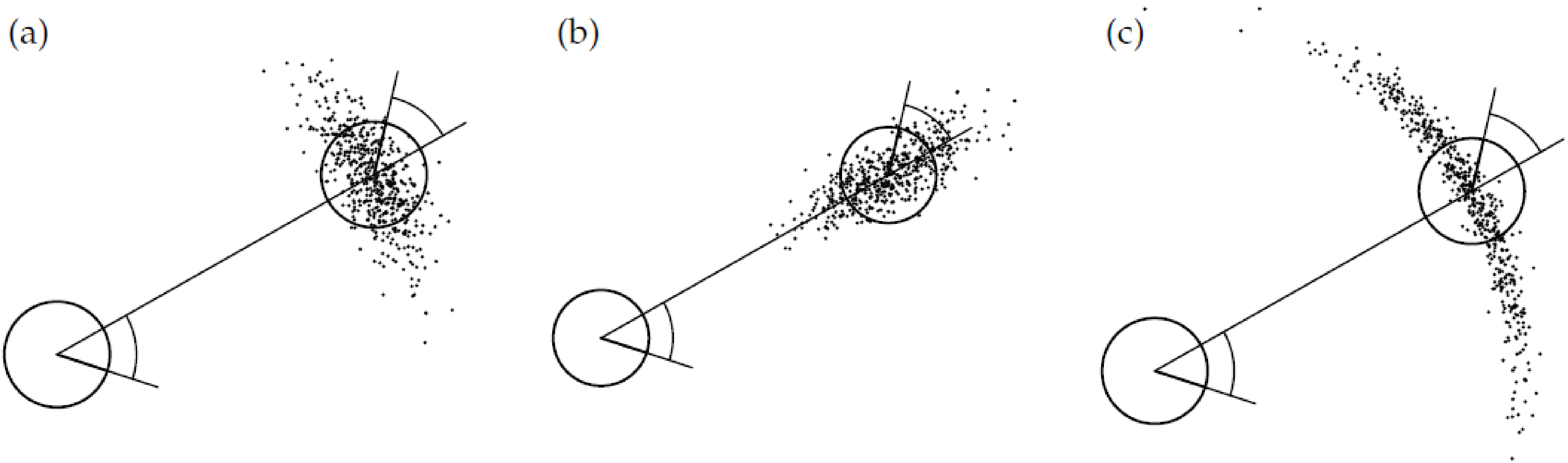
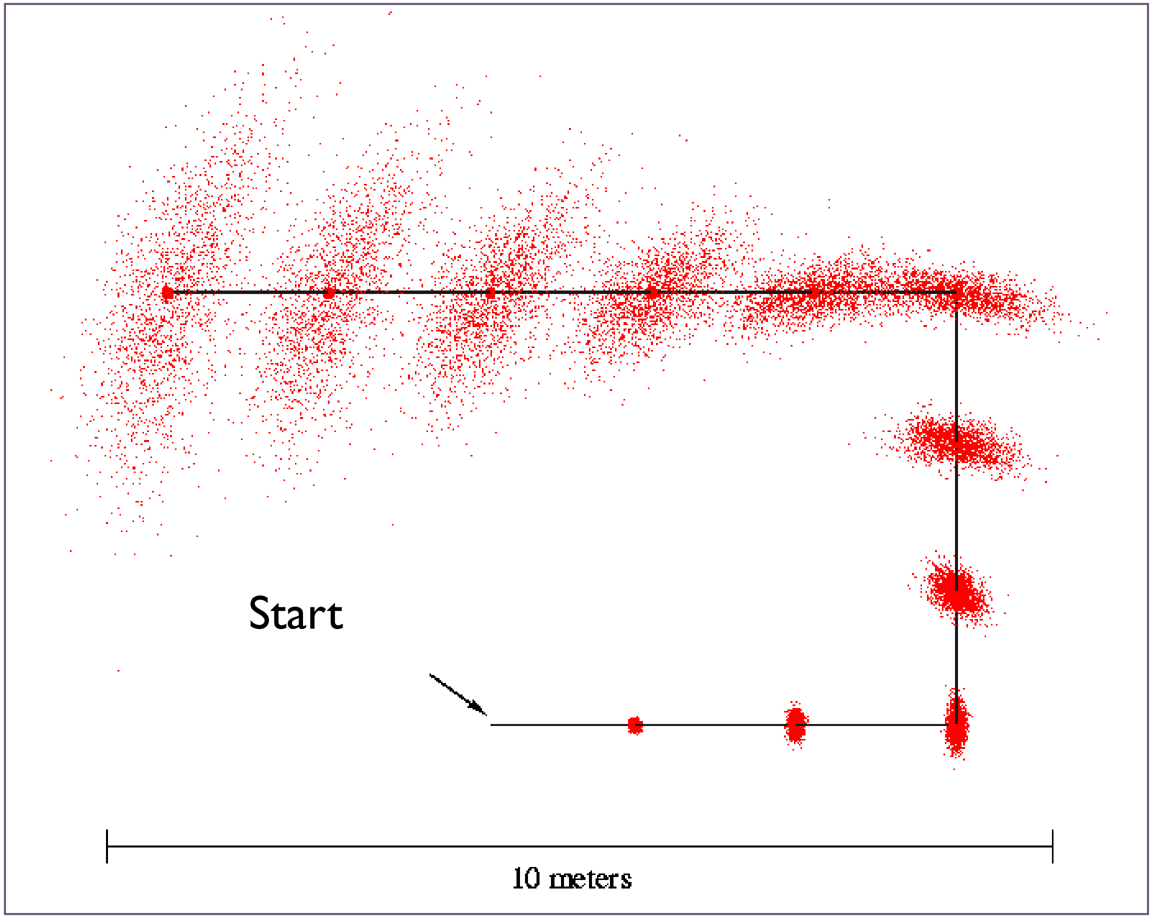


Figure 5.9 Sampling from the odometry motion model, using the same parameters as in Figure 5.8. Each diagram shows 500 samples.

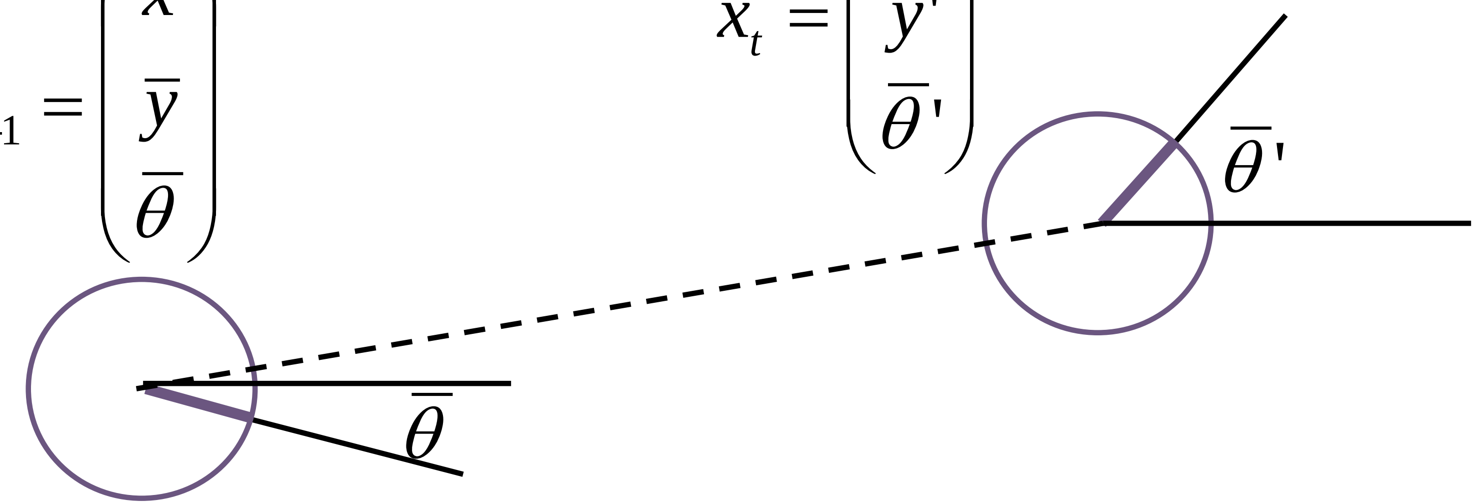
Sampling from Our Motion Model



Odometry Motion Model

- the key to computing $p(x_t \mid u_t, x_{t-1})$ for the odometry motion model is to remember that the robot has an internal estimate of its pose

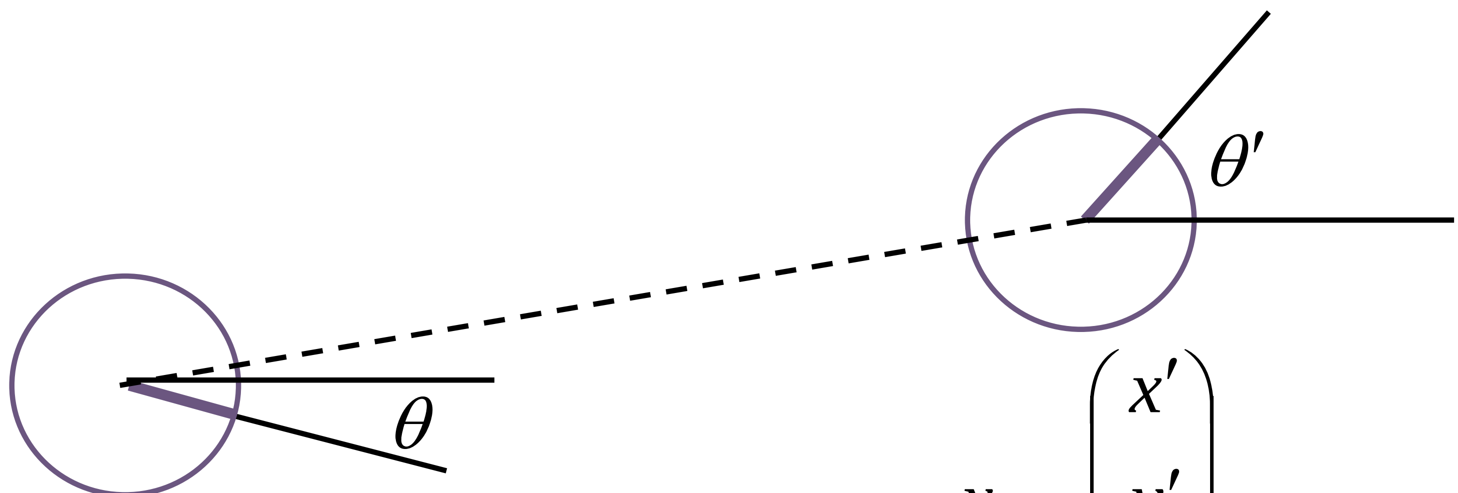
$$\bar{x}_{t-1} = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{\theta} \end{pmatrix} \qquad \bar{x}_t = \begin{pmatrix} \bar{x}' \\ \bar{y}' \\ \bar{\theta}' \end{pmatrix}$$



robot’s internal poses

Odometry Motion Model

- the key to computing $p(x_t \mid u_t, x_{t-1})$ for the odometry motion model is to remember that the robot has an internal estimate of its pose

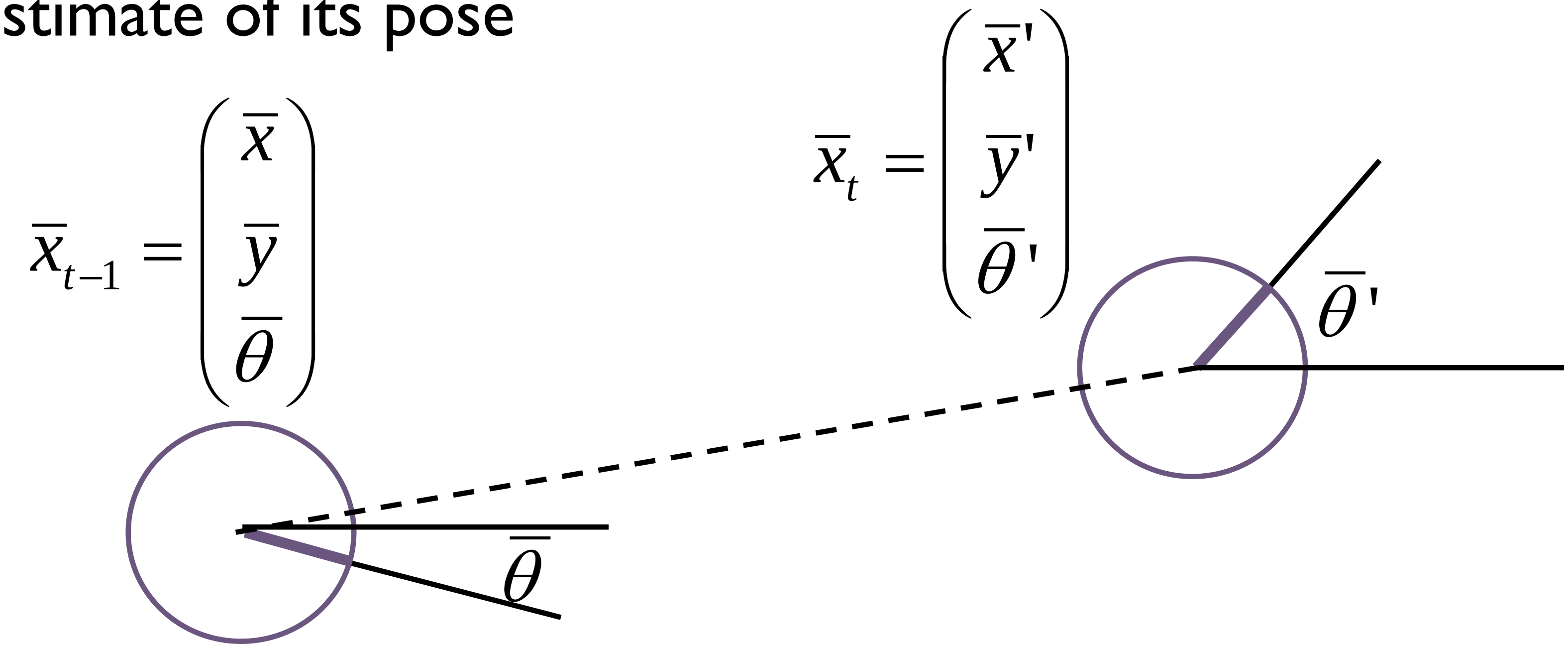


$$x_{t-1} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \qquad x_t = \begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix}$$

given poses

Odometry Motion Model

- the key to computing $p(x_t \mid u_t, x_{t-1})$ for the odometry motion model is to remember that the robot has an internal estimate of its pose



robot's internal poses

Odometry Motion Model

- the control vector is made up of the robot odometry

$$u_t = \begin{pmatrix} \bar{x}_{t-1} \\ \bar{x}_t \end{pmatrix}$$

- use the robot's internal pose estimates to compute the δ

$$\delta_{trans} = \sqrt{(\bar{x}' - \bar{x})^2 + (\bar{y}' - \bar{y})^2}$$

$$\delta_{rot1} = \operatorname{atan2}(\bar{y}' - \bar{y}, \bar{x}' - \bar{x}) - \bar{\theta}$$

$$\delta_{rot2} = \bar{\theta}' - \bar{\theta} - \delta_{rot1}$$

Odometry Motion Model

- use the given poses to compute the δ

$$\hat{\delta}_{trans} = \sqrt{(x'-x)^2 + (y'-y)^2}$$

$$\hat{\delta}_{rot1} = \operatorname{atan2}(y'-y, x'-x) - \theta$$

$$\hat{\delta}_{rot2} = \theta' - \theta - \hat{\delta}_{rot1}$$

- as with the velocity motion model, we have to solve the inverse kinematics problem here
  - but the problem is much simpler than in the velocity motion model

Odometry Motion Model

- recall the noise model

$$\delta_{trans} - \hat{\delta}_{trans} = \varepsilon_{\alpha_3 \hat{\delta}^2_{trans} + \alpha_4 (\hat{\delta}^2_{rot1} + \hat{\delta}^2_{rot2})}$$

$$\delta_{rot1} - \hat{\delta}_{rot1} = \varepsilon_{\alpha_1 \hat{\delta}^2_{rot1} + \alpha_2 \hat{\delta}^2_{trans}}$$

$$\delta_{rot2} - \hat{\delta}_{rot2} = \varepsilon_{\alpha_1 \hat{\delta}^2_{rot2} + \alpha_2 \hat{\delta}^2_{trans}}$$

which makes it easy to compute the probability densities of observing the differences in the δ

$$p_1 = \text{prob}(\delta_{trans} - \hat{\delta}_{trans}, \alpha_3 \hat{\delta}^2_{trans} + \alpha_4 (\hat{\delta}^2_{rot1} + \hat{\delta}^2_{rot2}))$$

$$p_2 = \text{prob}(\delta_{rot1} - \hat{\delta}_{rot1}, \alpha_1 \hat{\delta}^2_{rot1} + \alpha_2 \hat{\delta}^2_{trans})$$

$$p_3 = \text{prob}(\delta_{rot2} - \hat{\delta}_{rot2}, \alpha_1 \hat{\delta}^2_{rot2} + \alpha_2 \hat{\delta}^2_{trans})$$

Odometry Motion Model

1. Algorithm **motion_model_odometry(x,x',u)**
2. $\delta_{trans} = \sqrt{(\bar{x}'-\bar{x})^2 + (\bar{y}'-\bar{y})^2}$
3. $\delta_{rot1} = \text{atan2}(\bar{y}'-\bar{y}, \bar{x}'-\bar{x}) - \bar{\theta}$
4. $\delta_{rot2} = \bar{\theta}' - \bar{\theta} - \delta_{rot1}$
5. $\hat{\delta}_{trans} = \sqrt{(x'-x)^2 + (y'-y)^2}$
6. $\hat{\delta}_{rot1} = \text{atan2}(y'-y, x'-x) - \theta$
7. $\hat{\delta}_{rot2} = \theta' - \theta - \hat{\delta}_{rot1}$
8. $p_1 = \text{prob}(\delta_{\text{rot1}} - \hat{\delta}_{\text{rot1}}, \alpha_1 \hat{\delta}^2_{\text{rot1}} + \alpha_2 \hat{\delta}^2_{\text{trans}})$
9. $p_2 = \text{prob}(\delta_{\text{trans}} - \hat{\delta}_{\text{trans}}, \alpha_3 \hat{\delta}^2_{\text{trans}} + \alpha_4 (\hat{\delta}^2_{\text{rot1}} + \hat{\delta}^2_{\text{rot2}}))$
10. $p_3 = \text{prob}(\delta_{\text{rot2}} - \hat{\delta}_{\text{rot2}}, \alpha_1 \hat{\delta}^2_{\text{rot2}} + \alpha_2 \hat{\delta}^2_{\text{trans}})$
11. return $p_1 \cdot p_2 \cdot p_3$

odometry values (u)

values of interest (x,x')

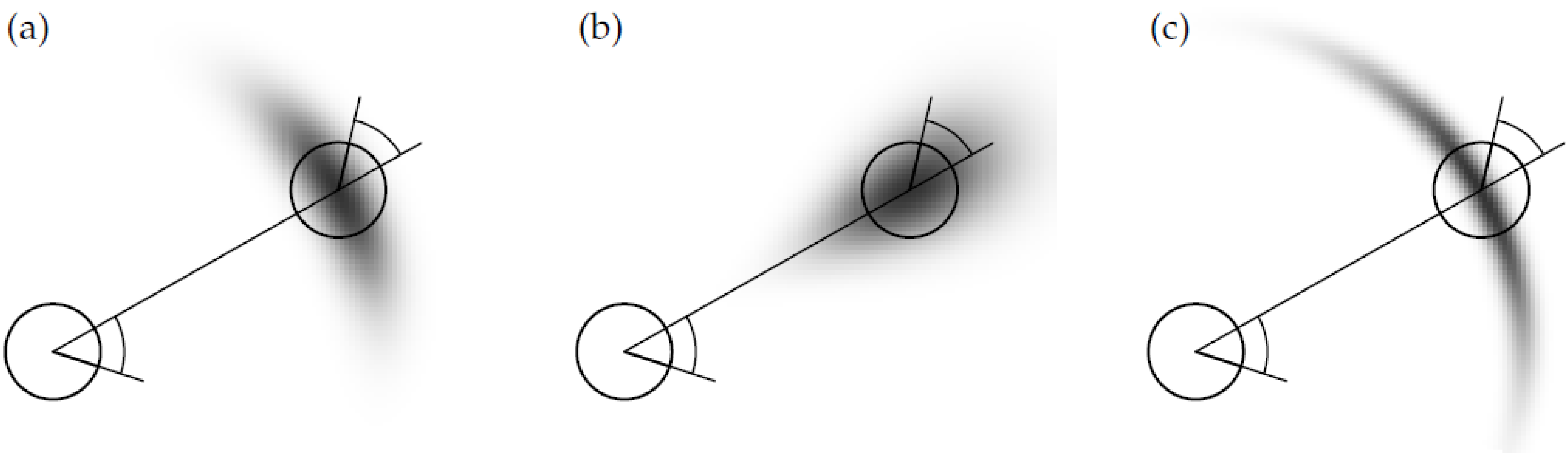
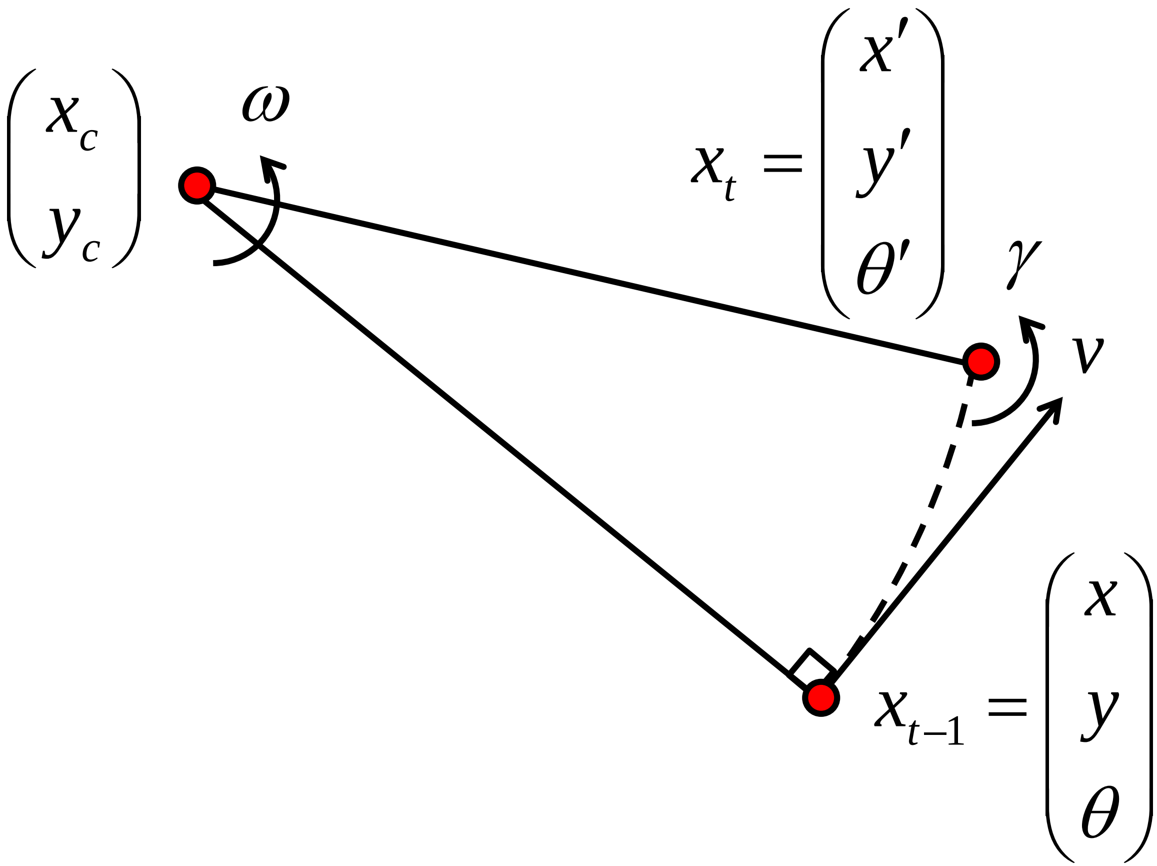


Figure 5.8 The odometry motion model, for different noise parameter settings.

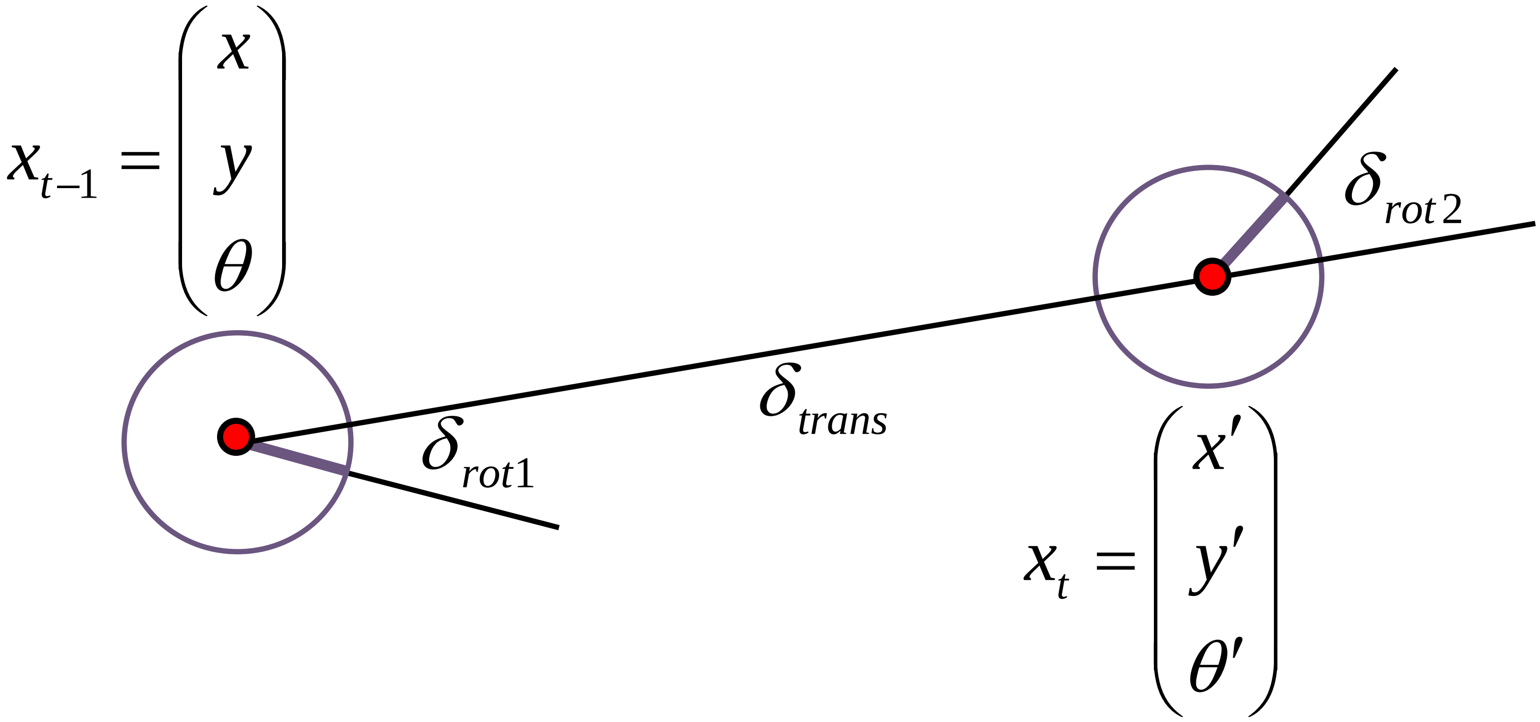
Recap

- velocity motion model
  - control variables were linear velocity, angular velocity about ICC, and final angular velocity about robot center



Recap

- odometric motion model
  - control variables were derived from odometry
    - initial rotation, translation, final rotation



Recap

- for both models we assumed the control inputs u_t were noisy
- the noise models were assumed to be zero-mean additive with a specified variance

$$\begin{pmatrix} \hat{v} \\ \hat{\omega} \end{pmatrix} = \begin{pmatrix} v \\ \omega \end{pmatrix} + \begin{pmatrix} v_{\text{noise}} \\ \omega_{\text{noise}} \end{pmatrix}$$

actual velocity | commanded velocity | noise

$$\text{var}(v_{\text{noise}}) = \alpha_1 v^2 + \alpha_2 \omega^2$$
$$\text{var}(\omega_{\text{noise}}) = \alpha_3 v^2 + \alpha_4 \omega^2$$

Recap

- for both models we assumed the control inputs u_t were noisy
- the noise models were assumed to be zero-mean additive with a specified variance

$$\begin{pmatrix} \hat{\delta}_{trans} \\ \hat{\delta}_{rot1} \\ \hat{\delta}_{rot2} \end{pmatrix} = \begin{pmatrix} \delta_{trans} \\ \delta_{rot1} \\ \delta_{rot2} \end{pmatrix} + \begin{pmatrix} \delta_{trans,noise} \\ \delta_{rot1,noise} \\ \delta_{rot2,noise} \end{pmatrix}$$

actual motion | commanded motion | noise

$$\mathrm{var}(\delta_{trans,noise}) = \alpha_3\, \hat{\delta}^2_{trans} + \alpha_4\, (\hat{\delta}^2_{rot1} + \hat{\delta}^2_{rot2})$$

$$\mathrm{var}(\delta_{rot1,noise}) = \alpha_1\, \hat{\delta}^2_{rot1} + \alpha_2\, \hat{\delta}^2_{trans}$$

$$\mathrm{var}(\delta_{rot2,noise}) = \alpha_1\, \hat{\delta}^2_{rot2} + \alpha_2\, \hat{\delta}^2_{trans}$$

Recap

- for both models we studied how to derive $p(x_t \mid u_t, x_{t-1})$
  - given
    - $x_{t\text{-}1}$ current pose
    - u_t control input
    - x_t new pose

  find the probability density that the new pose is generated by the current pose and control input
- required inverting the motion model to compare the *actual* with the *commanded* control parameters

Recap

- for both models we studied how to sample from $p(x_t \mid u_t, x_{t-1})$
  - given
    - $x_{t\text{-}1}$ current pose
    - u_t control input

    generate a random new pose x_t consistent with the motion model
- sampling from $p(x_t \mid u_t, x_{t-1})$ is often easier than calculating $p(x_t \mid u_t, x_{t-1})$ directly because only the forward kinematics are required